Al-Jabar: Jurnal Pendidikan Matematika
Vol. 12, No. 2, 2021, Hal 295 – 314
p-ISSN: 2086-5872, e-ISSN: 2540-7562

Metacognitive failure in constructing proof and how to scaffold it

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Article Info
Submitted : 29 — 07 — 2021
Revised : 26 — 08 — 2021
Accepted : 13 — 09 — 2021
Published : 15 — 12 — 2021

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Abstract

This research aims to describe the students' metacognitive failure in constructing proof and the scaffolding support. The participants of this qualitative case study were eight preservice mathematics teachers of six-semester, State University of Malang. We carried out a test about proof construction problems in Abstract Algebra. Then we verified the data using triangulation of constant comparative method from a test and a task-based interview with the stimulated recall. The results indicated two groups of students in proving strategy. Group I performed "appropriate" syntactic strategy and Group II vice versa. **Blindness** was experienced by the subject that does not recognize errors detection or the ambiguity of the proof. **Mirage** occurred when the subject recognizes an error detection on the proper strategy or application of a theorem, then is unable to verify the truth of his work. **Misdirection** appeared when the subject recognizes a lack of progress, then uses an incomplete or irrelevant concept. **Vandalism** emerged with no progress or detection of errors of the strategy then the subject performs some irrelevant steps to the issue or uses a misconception. Practically, the teachers can use these results for learning innovations in scaffolding-based proof courses. The scaffolding might need some development and application in supporting students to overcome difficulty in proving mathematical sentences.

Key Words: Constructing Proof, Metacognitive Failure, Scaffolding

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Introduction

The proof is one of the important things in Mathematics, as proof becomes the basis in mathematical activities (Hanna, 2018; Sirmaci, 2012; Wittmann, 2021; Zengin, 2017). The validity of the theorem in mathematics can be demonstrated by the existence of proof (CadwalladerOlsker, 2011; Ozan et al., 2021). Furthermore, proof and reasoning play important roles to show the truth of the solution of mathematical problems in learning mathematics (Hamami & Morris, 2020; Varghese, 2009; Wittmann, 2021). The ability to construct a proof for mathematicians, mathematics teachers, and mathematics students becomes one of the important things and as an assessment of student performance in learning advanced mathematics such as abstract algebra and real analysis (Moore, 2016; Thomas et al., 2015; Wasserman et al., 2018).

The process of constructing the proof can be seen as a process of mathematical problem-solving (Hamami & Morris, 2020; Nunokawa, 2010; Weber, 2001; Zimmermann, 2016). Problem-solving strategy is often influenced by the knowledge and skill of an individual in obtaining proof solutions (Hughes et al., 2019; Weber, 2001). Problem-solving activity is closely related to cognitive and metacognitive activity. Metacognitive skills are important in all mathematical performance, including problem-solving (Anggoro et al., 2019; Chytrỳ et al., 2020; Garofalo & Lester Jr, 1985; Magiera & Zawojewski, 2011; Zhao et al., 2019).

The main objective of the metacognitive process is that people are still thinking on the right track of solutions (Ishikawa et al., 2019; Puente-Díaz et al., 2021). Through the role of metacognitive activity, someone can prepare and achieve goals to get a solution for the problem-solving process, as well as improving the quality of the proof solution (Barbacena & Sy, 2013; Biryukov, 2014). The existence of metacognitive failure can lead to several factors that influence problem-solving, such as the detection of an error (error detection) in the process of problem-solving, lack of progress in the process of finding a solution, and anomalous results. These three factors are often called the "*red flag*" (Goos et al., 2000; Goos, 2002; G. Stillman, 2011). Metacognitive failure in problem-solving can lead solvers to get inappropriate solutions. The metacognitive "*red flag*" shows an indication of the necessity to stop or re-examine someone's process of problem-solving due to certain difficulties experienced (Goos, 2002; Huda et al., 2016; Stillman, 2011). Common types of metacognitive failure are metacognitive blindness, metacognitive mirage, and metacognitive vandalism (Goos, 2002). Furthermore, two other types of failure namely metacognitive misdirection and metacognitive impasse (Stillman, 2011; Stillman, 2015).

In some research, metacognitive activities are not explained in detail when metacognition that recorded leads to error in problem-solving (Alifiani & Walida, 2020; Magiera & Zawojewski, 2011), then further research has been carried out regarding metacognitive failure at the stage of problem-solving type eliciting activities model (Rozak, 2018). Other research reveals a person's metacognitive failure when solving problems independently (Huda et al., 2018; Oliviani, 2018), as well as the process of one's metacognitive success in groups, and not focusing on metacognitive failure (Goos, 2002). The three metacognitive failures called blindness, mirage, and vandalism in problem-solving proof have been investigated through the assimilation and accommodation framework (Huda et al., 2016). Meanwhile, metacognitive failure at each stage of proving activity has not been widely studied. In addition, when and what support can be given while a person experiences a metacognitive failure needs further research.

A cooperative effort by someone who has more knowledge with learners to solve problems and the learners will be able to complete their works by providing support, namely scaffolding (Pol et al., 2019; Reiser, 2004; Van Der Stuyf, 2002; Wright, 2018). This support facilitates the learners to rebuild the prior knowledge and acquiring new information then they succeed to outgrow their problem-solving difficulties (Basir & Wijayanti, 2020; Margulieux & Catrambone, 2017; Reiser, 2004). Scaffolding refers to the process by which the support provided to learners is gradually reduced to counter the side effects of excessive problem-solving complexity (Anghileri, 2006; Kilic, 2018; Könings et al., 2019; Salem, 2019; Shvarts & Bakker, 2019; Wood et al., 1976; Zackariasson, 2019). Thus, the process of scaffolding holds potential alternatives for students to overcome metacognitive failure in the problem-solving process, particularly in constructing proofs.

This research determines the process of students' metacognitive failure to construct proof and the scaffolding. The metacognitive failure was assessed based on five components, namely blindness, mirage, misdirection, impasse, and vandalism. The metacognitive failure process is used to catch the failure in constructing the proof and used to determine the appropriate form of scaffolding. Therefore, discussion about metacognitive failure and scaffolding become an important part to overcome students' difficulties in constructing proofs.

The Research Methods

This research was a case study, conducted to explore the failure of the metacognitive process experienced by students in proof construction, along with the scaffolding that can be done to overcome it. The proving problem used in this research relates to the theorem in group theory in abstract algebra. The research was conducted at the State University of Malang. The subjects involved in this research were eight students of the sixth semester of preservice mathematics teacher. They were selected based on the recommendation of the teacher of Abstract Algebra and their communication ability.

The test instruments consist of three problems with different types. Problem 1 related to the definition of the cyclic group and abelian group in Abstract Algebra. Problem 2 deals with definitions and theorems to the group theory as well as the order in abstract algebra. Problem 3 deals with the strategies used to choose an existing definition or theorems. Then, the subject did "appropriate" syntactic strategies (called group I) consisted of three people and was presented by two persons (S1 dan S2). Furthermore, the subject did an "inappropriate" syntactic strategy (called group II) consisted of five people and was explained by their representatives of two persons (S3 dan S4). Syntactic strategy is performed in constructing the proof by manipulating the given definition and mathematical facts that are logically available corresponding rules of inference in the mathematical system (Fatmiyati et al., 2020; Weber & Alcock, 2004).

Work instructions:
Do the following tasks in detail and clearly

1. Suppose G is a cyclic group with generator a. Prove that G is abelian
2. Suppose G is a finite group and $a \in G$. Prove that the order of a and a^{-1} has the same order
3. Let ϕ is a homomorphism from G group to G' group, and H is the subgroup of G. Prove that $\phi(H)$ is a subgroup of G'.

Figure 1. Test Item Instrument

Data collected through tests and in-depth interview task-based, with the stimulated recall, which was one method of data collection that can be used to investigate the cognitive process and decision-making of subject by showing the sequence of events in the video or other forms of visual recall (Denley & Bishop, 2010; Falloon, 2020; Geiger et al., 2018). The work of students and task-based interviews are analyzed using a constant comparative method to illustrate the metacognitive failure and the appropriate scaffolding. The data collected from test and task-based interviews were validated using a constant comparative method to illustrate the metacognitive failure and the appropriate scaffolding. The complete research method flow as Figure 2.

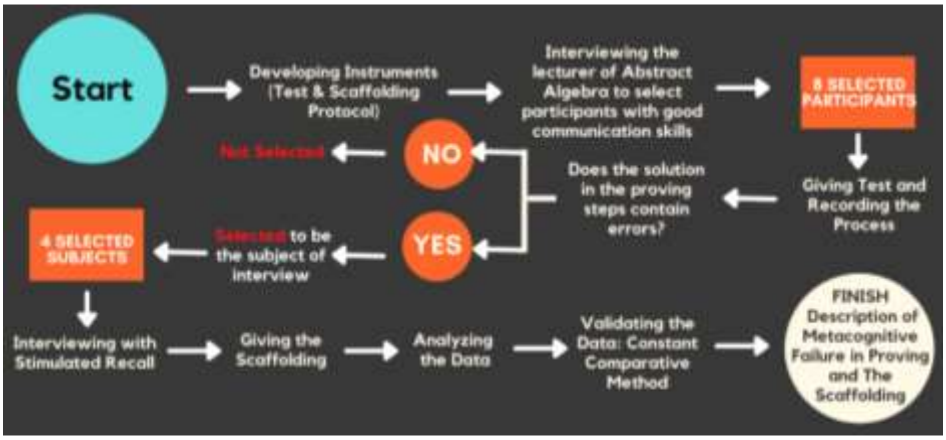


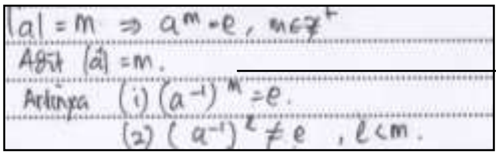
Figure 2. The Research Method Flow

The Results of the Research and the Discussion

We determine the process of metacognitive failure in constructing the proof. In all groups of subjects, metacognitive failure emerges when constructing proofs of problems in the given test, we resume it in Table 1. One example of the proof construction of the subject group I, Figure 3, subject using the axiom that G is a finite group with $a \in G$. Then proceed by using a mathematical fact about the concept of the equality of two numbers and the corollary of order. Furthermore, in Figure 4, one example of the proof construction of the subject group II that using the axiom that G has n element, but followed by letting $a = a^{-1}$, that is incompatible with the mathematical fact that related to the statement and not appropriate to show the similarity of order. The proof construction process on both of them involved syntactic strategy because there is a form of manipulation definition or a mathematical fact in symbols. From the proof construction process obtained in group I, it tends to do the "appropriate" syntactic strategy, while the second group tends to do “inappropriate”.

In Figure 3, the object cannot perform the "appropriate" syntactic strategies when showing the order of two elements in the group were the same. This was proven from the finding of the interview with the subject, as follows.

S_3: I thought that order of a was the smallest positive number. The finite group [the number] of its elements. So that I have to know how many elements of G, to be able to define the order of a. The number of elements of G [there is] n element, that was the order that I thought [order of a and a^{-1}] would be the same with n, so I have to write first if $a = a^{-1}$.



It shows that order a is equal to a^{-1} with $|a^{-1}| = m$, where m is an order from a, then applies for corollary order.

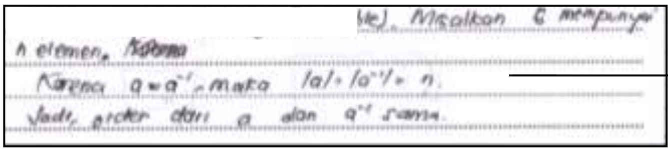
Translation:
$|a| = m \Rightarrow a^m = e, m \in \mathbb{Z}^+$
We'll show $|a^{-1}| = m$
It means (1) $(a^{-1})^m = e$, (2) $(a^{-1})^l \neq e, l < m$

Figure 3. One of the Example of “Appropriate” Syntactic Strategy

Table 1. The Coded Metacognitive Failure

No	Group	Subject	Task	Red Flag	Failure	Scaffolding
1	I	S_1	1	No progress and impasse solution	Vandalism	Level 2 *reviewing*, Level 3
			2	The incomplete concept use	Misdirection	Level 2 *reviewing* and *restructuring*
			3	Error detection on right strategy	Mirage	Level 2 *reviewing* and *restructuring*
2	I	S_2	1	Error detection on right strategy	Mirage	Level 2 *reviewing* and *restructuring*
			2	Error detection	Blindness	Level 2 *reviewing*, Level 3
			3	Error detection	Blindness	Level 2 *reviewing*, Level 3
3	II	S_3	1	Anomalous result; error detection	Blindness	Level 2 *reviewing* and *restructuring*
			2	Error detection on right strategy	Mirage	Level 2 *reviewing* and *restructuring*
			3	Relevant concept but appropriate	Misdirection	Level 2 *reviewing* and *restructuring*

No	Group	Subject	Task	Red Flag	Failure	Scaffolding
4	II	S_4	1	No progress, but use inconsistent concepts	Vandalism	Level 2 *reviewing* and *restructuring*
			2	Error detection and no progress in the work steps, use inappropriate concepts	Vandalism	Level 2 *reviewing* and *restructuring*
			3	The incomplete concept use	Misdirection	Level 2 *reviewing* and *restructuring*



It shows that order a is equal to a^{-1}.
Subject assumes that $a = a^{-1}$

Translation:
Let G has n elements.
Since $a = a^{-1}$, then $|a| = |a^{-1}| = n$
So, the order of a and a^{-1} are same

Figure 4. One of the Example of "Inappropriate" Syntactic Strategy

Metacognitive failures arise when someone's metacognitive activity leads to error answers and solutions in the process of problem-solving (Goos, 2002; Stillman, 2011). There are three metacognitive activities, first, *metacognitive awareness*-**MA**, when one realizes to think about the position of his knowledge during the process of problem-solving, what strategies is needed, and can be done in the context of problem-solving, as well as the relationship between knowledge possessed by a strategy that can be used. Metacognitive awareness is also a metacognitive activity in which a person is aware to think about what he knows as well as his ability in problem-solving. Furthermore, *metacognitive evaluation*-**ME** leads to considerations related to the person's thought processes, one is aware of the limitations and to think about the effectiveness of the knowledge and ability in problem-solving, the effectiveness of the chosen strategy, assesses the level of difficulty of the problem, and assesses the results of problem-solving. Then the third is *metacognitive regulation* -**MR**, someone using his cognitive resources or rethink about what he was thinking to plan, define work steps along with the purpose of each step of the work done, choose and plan the most appropriate strategy, as well as prioritizing and selecting step of appropriate work (Magiera & Zawojewski, 2011). From the results of diagnostic tests and the interview in both groups of subjects appear metacognitive failure, namely *blindness*, *mirage*, *vandalism*, and *misdirection* accompanied by metacognitive activity MA, ME, and MR.

Metacognitive Blindness

For Subject Group 1, S_2 experienced metacognitive *blindness*, especially in tasks 2 and 3. Metacognitive blindness occurred with a *red flag*: cannot recognize error detection. On task 2, S_2 was not able to realize the error, namely the assumption on indirect proof and also the subsequent steps. Furthermore, on task 3, S_2 had misconceptions about subgroups that contain only a subset concept, then the misconceptions of homomorphism were not recognized by her. S_2 decided to use proof by contradiction in one part of the statement in task 2, which m be the smallest positive number such that $(a^{-1})^m = e$. S_2 assumed $l = m + s$ and showed $(a^{-1})^l \neq e$. One example of the failure of metacognitive *blindness*, from the stimulated recall of S_2, shows that she remained unaware of *red flag* error detection in the assumptions she made and inferences that appear on the proof by contradictions. Therefore, S_2 encountered blindness, S_2 did not realize the *red flag* error detection (Goos, 2002). This result about making a statement $l = m + s \geq m$ become a *red* *flag* error was not recognized in the proof problem (Huda et al., 2016). The discoveries of constructed proof, stimulated recall, and interview of S2, are as follows in Figure 5.

Translation:
$(a^{-1})^l = e$ because $l < m$.
Assumption $l = m + s \Rightarrow a^l = e = a^{m+s}$
$(a^{-1})^l = (a^l)^{-1} = (a^{m+s})^{-1} = (a^m \cdot a^s) = (e \cdot a^s)^{-1} \neq e$ contradiction
So, $(a^{-1})^l \neq e$.

Figure 5. *Red flag* of S2

S2: ...In my mind, this would indicate that a *with the power of something is not equal identity, e(MA), for [the power] something less than* m ***-Obj**, hmm it means I want to use contradictions—***Cd** *[MR]. I assume for example that* l, $l = m + s$*—***A1** ***(RED FLAG—*****Rf1),** *means that* a^{m+s} *is equal with* e ***-A2** *[MR]. Then I think again, oh yes, the result there is a contradiction that is not equal [with e] (MR).*

R: Try to explain (showing a video while S2 does task 2), have you ever made a mistake, and then you check it again?

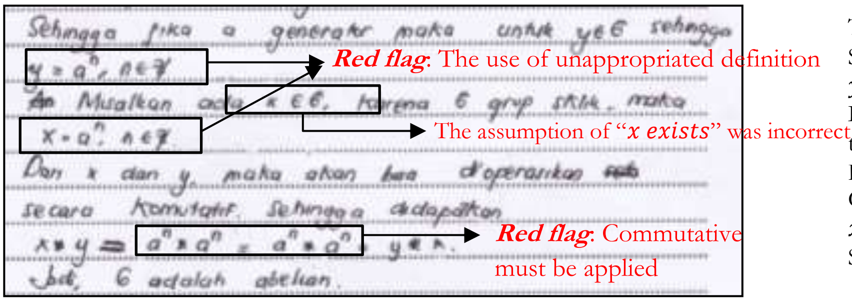
S2: Yes, I did [I thought I was wrong] (MA), I thought it was like this (pointing $[e \cdot a^s]^{-1} \neq e$*—***B1***—**RED FLAG*** **—Rf2***) to be able to know directly (MA), oh it is not equal (MR), then I tried to think again and wrote that supposed to show that contradicts to the facts —***A2** *(MA), [I think] is it allow to be like that? or not? (ME). It might be right (ME), the thing that was in my thought finally shown [in the way] like this (MR).*

For a subject of group II, S3 encountered *blindness* on task 1. He did not recognize the *red flag* ambiguous of the final answer on the result, i.e., $a^n * a^n = a^n * a^n$. Metacognitive activities of S3 were not successful in evaluating the final answer with commutatively guaranteed. We explored the S3's thinking process when he defined x and y, as a^n for $n \in \mathbb{Z}$. A series of S3's metacognitive activities showed he did not recognize *red flags* on the use of error detection on definitions. In addition, the ambiguity of the final statement $a^n * a^n$ could be recognized as a *red flag* by S3 and ignored. The discoveries of the interview with S3 are as follows.

R: Tell me what did you think about what were the steps that you did (pointing to a video) and also what did you think before taking those steps?

S3: I thought again, was it right for any x*, I can get* x again which was equal to a^n **–B1***(ME). The problem was that possible or not if* $x * y$ *became* $a^n * a^n$ **–Obj2** ***(RED FLAG*** **–Rf3***) (ME). I doubted it though (MA). I thought again from the definition [generator* **—Gen***] maybe it did exist (ME). So I was able to take* x *(MR). So that I needed to write again the definition of* x **–B1***(MA). If I took it instead of taking* y*, but* x *(MA) because the generator is* a*, [every element is]* a^n *where* n *is integers. So it meant that from that two definitions (MA) the generator could be taken if* $y = a^n$, $x = a^n$ ***(RED FLAG*** **–Rf4***) (MR, and when those were operated the result were the same (MA), those were things that I thought before writing these steps.*

The following Figure 6 shows S3 performed proof of task 1 such that he encountered *blindness*. Under *blindness* when metacognitive activities do not bring their "error detection" and skip the errors made (Goos, 2002; Stillman, 2011). However, an emerging new *red flag* that was not recognized in the form of "anomalous result", then errors in the work steps remain to be done.



Translation:
So, if a is a generator, then for $y \in G$ therefore $y = a^n, n \in \mathbb{Z}$
Let there is $x \in G$, because G is a cyclic group, then $x = a^n, n \in \mathbb{Z}$.
From x and y, then it can be operated Commutatively. So it becomes
$x * y = a^n * a^n = a^n * a^n = y * x$
So, G is abelian.

Figure 6. *Red flag* of S_3

We relate to other researches, which uses problem-solving to explore metacognitive failures, problems can provide access to students for an outcome that serves as a reference when solution or calculations error performed (Goos, 2002). Therefore with the access students can prepare themselves for the possibility of errors in their working steps (Stillman, 2011). While in this research, the access that the subjects could realize their errors depends on the knowledge of subjects related to the given mathematical proof problem. So, the errors that appear in this research was not calculation error, but the use of the concepts and properties, especially in constructing proofs. A subject experienced an error after metacognitive *awareness* and metacognitive *regulation*, in considering the outcome, but another subject made an error of metacognitive *evaluation*. It shows a reflection of distinction but also equality with other discoveries (Huda et al., 2018).

Scaffolding was performed on S_2 (in tasks 2 and 3) in the form of *reviewing* by requesting S_2 rethink that statement proved or strategies used. Similar to the previous research that scaffolding has characteristics in verifying and clarifying students' understanding, by asking what the meaning of the problem and what are the students have to do (Basir & Wijayanti, 2020; Gusmiyanti et al., 2018; Pol et al., 2019). After S_2 was able to realize the errors then scaffolding continued to level 3 (**L3**); that is a discussion about improving working steps (**A'**) carried by S_2 and a discussion of the reasonings why do or do not a certain step. This is the final step of scaffolding by conducting asks and answers to explore the difficulties experienced by students and lead to the correct answer (Basir & Wijayanti, 2020). One example of scaffolding and the improved proof was provided by S_2 in Figure 7, as follows.

*R : Try to think again and explain what were the statements that you tried to prove in this second? **(Rev)***
S_2: For any $l < m$, so $a^l \neq e$.
R : From those steps that you have done, what kind of proving that you have used?
S_2: Contradiction. This [statement] was assumed as incorrect. If this is incorrect….(mumbling unclearly)
R : Please try to think again about the statement that you tried to prove.
S_2: Hmm, so it means that what should be assumed (pointing on $(\forall l < m)\ (a^{-1})^l \neq e$), that are going to be shown, assumed incorrectly. Yes. It means I should write, assumption $a^l \neq e$ for any l less than m. (mumbling)
R : Try to think again about the strategies you have done.
*S_2: This is a wrong assumption **(A1)** that was incorrect.*
*R : Why was it incorrect? **(L3)***
S_2: I assumed $l = m + s$ if I did so, the l would not be clear then, s would not be clear too (laughing)
R : Please try to think again, according to you, was there something that you need to fix?

S₂: Yes. Using contradiction. Let me redo it again, please. If I am not mistaken… (writing). Ohh this one…. a… uhm the other way. a^l … power m.. inverse. After that, a^m is the same with e (mumbling). So it means that a^l, uhm (stops for a while). Ohh wait, a^l if it is assumed as... Ohh $(a^{-1})^m$ is also an information right. So this… uhhm,.. so a^l… Ohh right, it shows that ($a^l = e$) (**A'**). *It means contradiction happens.*

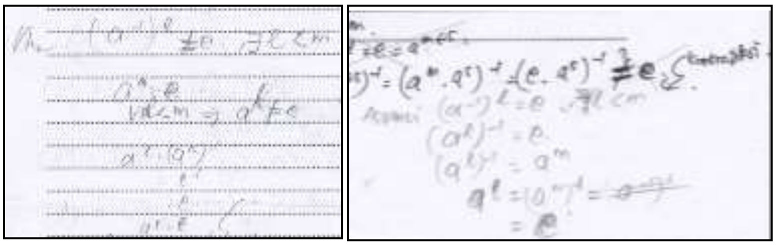


Figure 7. Improved Proof - S_2

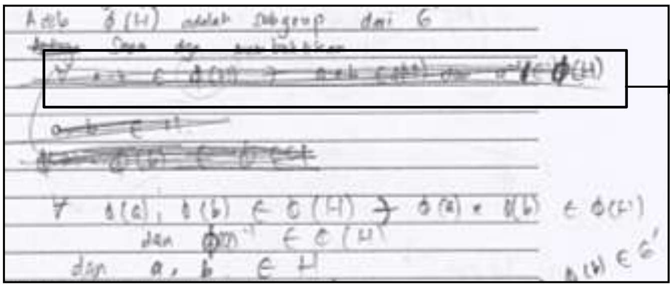
In task 3, *level 3* was done by discussing the definition of subgroups, that are not in line with the construction of the proofs provided. We provided a light reinforcement of the steps done by S_2. We can connect to term contextualization, draws new knowledge closer by creating a new intermediate level of representation to connect the concepts introduced with others that students build direct experience (Basir & Wijayanti, 2020).

Scaffolding for S_3 was done in *reviewing* by asking to evaluate whether the final solutions are appropriate. Next, *restructuring* carried out in S_3 is to provide counterexamples of the given assumption such that S_3 was aware of errors that had been made (Anghileri, 2006). Subjects in group I were able to continue up to level 3, as they realized the error when *reviewing* held.

Metacognitive Mirage

Subjects in group I experienced metacognitive *mirage*, S_1 on task 3, while S_2 on task 1. S_1 and S_2 recognize *red flag* error detection on the application of the theorem on the metacognitive activity process done, although originally performed appropriately. On the subject of group II, S_3 experienced metacognitive failure on task 2 because he ignored the appropriate strategy. S_3 recognized *red flags* errors and was unable to give a decision on the validity of his work. Both of the metacognitive failure phenomena are in line with the mirage of other's discoveries (Goos, 2002; Huda et al., 2016; Stillman, 2011).

One example of metacognitive *mirage* that appeared in group I, when S_1 applied the theorem that was originally written correctly about any two elements of $\phi(H)$, but S_1 changed and made it false, become two elements in H. S_1 initially proved by assuming a and b were members of $\phi(H)$ then indicated the character of the closure and the inverse contained in $\phi(H)$. We conclude that S_1 experienced metacognitive mirage because S_1 misjudged the pace of his work. The following are S_1's statements when we explore metacognitive mirage possibly occurs through stimulated recall and also the work of S1 containing errors in Figure 8.



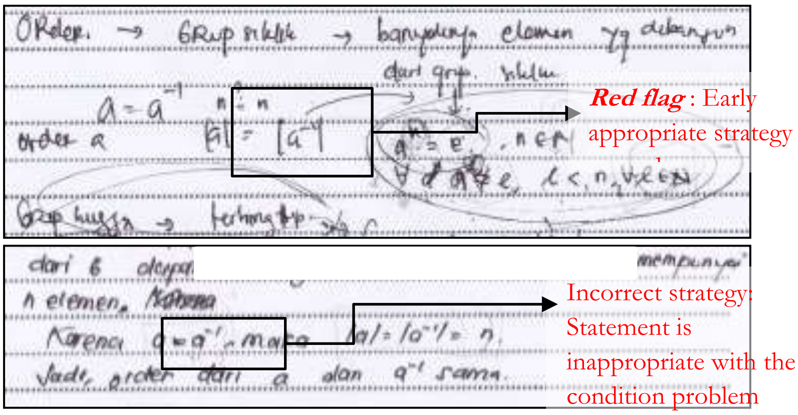
Red Flag: Removing the Right Step

Translation:
It'll show $\phi(H)$ is a subgroup of G
~~It means~~ It same with proving
$\cancel{\forall a,b \in \phi(H) \ni a*b \in \phi(H)}$ and
$\cancel{a^{-1} \in \phi(H)}$
$\cancel{a*b \in H}$
$\cancel{\phi(a)*\phi(b) \in \phi(H)}$
$\forall \phi(a),\phi(b) \in \phi(H) \ni \phi(a)*\phi(b)$

Figure 8. One Example of The Right Steps but Abandoned - S_1

*S₁: First I thought, a dan b were members of $\phi(H)$ –***C1** *(MA), so $a * b$ was also a member of $\phi(H)$ —***Obj3** *(MR). However, when I thought about it again using the function —***D** *(MR), proved that the members of $\phi(H)$ were not a and b (MA) but $\phi(a)$ and $\phi(b)$—***Im***. a and b were members of H –***C2** *(MA). So I thought again and reconsider it (ME). The inappropriate thing was when I take any a, b which were members of $\phi(H)$* ***(RED FLAG–Rf5)*** *(MA). Then I think again that it was wrong (ME-MA). So that I redo it. (MR).*

Another example of a metacognitive *mirage* that appeared on subject group II is when S_3 made an error by using inappropriate assumptions related to the problem. Initially, S_3 recognized the *red flag,* there was no progress on the working steps on the right strategy, which showed if the order of a is n, then the order of a^{-1} is n. From the undertaken metacognitive activities, S_3 ignored the strategy, then related to the problem with the definition of the finite group, then used another inappropriate strategy, namely presupposing $a = a^{-1}$ and the illogical conclusion that $|a| = |a^{-1}| = n$. In this case, S_3 experienced a *mirage* failure. The interviews with metacognitive activities that accompanied the metacognitive failure of S_3 and his error in Figure 9, as follows.



Translation:
Order → Cyclic Group → the number of the generated element from the cyclic group
$a = a^{-1}$ $n = n$
Order of a, $|a| = |a^{-1}|$, $a^n = e, n \in \mathbb{N}$
$\forall a^l \neq e, l < n, \forall l \in \mathbb{N}$
Finite group → countable
Let G has n elements.
Since $a = a^{-1}$, then $|a| = |a^{-1}| = n$
So, the order of a and a^{-1} are same

Figure 9. Inappropriate Strategy Abandoned - S_3

*S₃ :... If the order a was n can be written two things [corollary order –***Co***] (MA), order a^{-1} can be shown as the equal of n too–***D1***(MR). But I thought hard to show* ***(RED FLAG—Rf6)****, I don't know yet whether it is right or not (ME)... Evidently from the finite group definition and $a \in G$, there is a relation with a—***Fg** *(MA), but for a dan a^{-1}, they are just the same and I think again if [a dan a^{-1}] di G is equal to a and the inverse –***D2** *(ME). Then I conclude that is order a is equal to the order of a^{-1} directly —***Obj4***(MR). I thought the order a and a^{-1} will be equal to n (MA). In the other words, first I need to write if $a = a^{-1}$ –***D3** *(MR). But there is one step that I do not sure about in the part that concludes a is equal with a^{-1} (ME). Right or wrong the reason I have not known yet (MA).*

The previous research has found the importance of clarification and validation of the new ideas from metacognitive awareness that has the potential to support the problem-solving process. The use of metacognitive awareness can be known by the students in group interaction (Goos, 2002). In this research, subjects construct proofs individually, but they need to do a similar way to handle the metacognitive *mirage*. On the other hand, the subject is unable to evaluate the effectiveness of their own strategy, in line with the previous research that found the subject's failure in considering the effectiveness and limitation of their thinking (Huda et al., 2018).

The scaffolding on the level of *reviewing* was performed on S_1 and S_2. First, asking subjects to re-explain what step that they were judged wrong the reason for replacing with another step. The next level of *restructuring* has given by providing statements in the context of a simple and pointing to the truth of the step that S_1 was started. While on S_2, by providing reciprocal questions related to the truth of the proof in a simpler context. This discovery is supported the previous research about the teacher provided a lot of regulation by asking steering questions and some suggestions (Pol et al., 2019). The given scaffolding to S_2 has been continued up to level 3 which was comparing two constructions proof obtained as the connection to develop its strategy. These findings are in line with previous research (Anghileri, 2006). In group II, scaffolding has been given similarly.

One example of scaffolding in S_3 to overcome the metacognitive mirage in the form of *reviewing* was asking S_3 to explain the reasons for not using the right strategy. The chain of scaffolding, as follows.

R : While you finishing this task, what was the strategy you had been thinking about?
S_3: Oh, yes. I had been thinking about this strategy that I wrote, and the second strategy was a with order n, then I want to show the order of a^{-1} is n. But I was a bit confused with the second strategy
R : Uhmm, Can you explain more about your reason for not using the second strategy? ***(Rev1)***
S_3: I could not relate from the order a is n, more to the proving progression, how to show the order a a^{-1} became n. Finally, I tried another strategy.

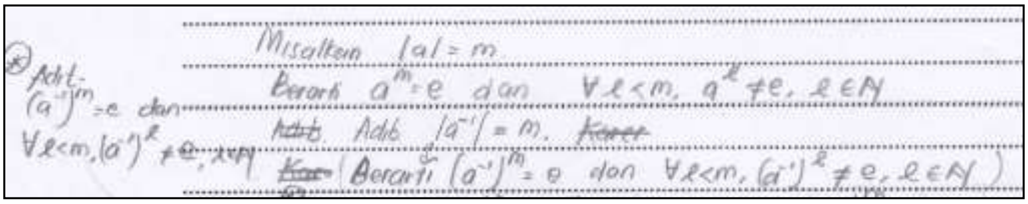
Furthermore, we continued to *restructuring* by asking S_3 to observe proof construction that already exists and ask questions about the validity of abandoned the strategy. We called correction feedback, when students make statements that are factually inaccurate or use the term in an inaccurate manner, the teacher offers information to clarify the truth which is actually inaccurate (Basir & Wijayanti, 2020). Here scaffolding was conducted to S_3.

R : Please, try to observe again this proof construction results. What kind of strategy do you think that can be applied according to the theorem or definition that you have? ***(Res1)***
S_3: The first one that I thought of, but because I found difficulties, I tried another one.
R : Well, try to look again at your construction proof, is it inline or not with the theorem or the definition that you have?
S_3: (mumbling) Sure not, this one does not have a theorem [that supports it].

The next *restructuring* has been done by providing a simple statement in a simple context and has the same characteristics. This leads to the truth of the earlier strategy, which showed up $|a^{-1}| = m$ with m is $|a|$, and how these strategies can be applied. Scaffolding is carried out through guidance in determining what strategies students should do in observing and manipulating some objects, discussing various mathematical concepts contained in an observed simpler context (Basir & Wijayanti, 2020). Here is the scaffolding that has been done to S_3 and the improved proof in Figure 10.

R : Suppose that the order a^{-1} is equal to 4. What do you think to solve it? ***(Res2)***
S_3: Uhmm, change it into corollary $(a^{-1})^4 = e$ and if it is powered to 1,2,3 is not equal to e.
R : Okay, now let's go back to the main problem, what is your purpose?
S_3: To prove that the order a is equal to $|a^{-1}|$.
R : Well, then if you use your previous strategy, how is it?
S_3: Uhmm, shown that order a^{-1} is m.
R : What is m?

S_3: The order of a.



Translation:

⊛
We will show,
$(a^{-1})^m = e$ and
$\forall l < m, (a^{-1})^l \neq e, l < m$

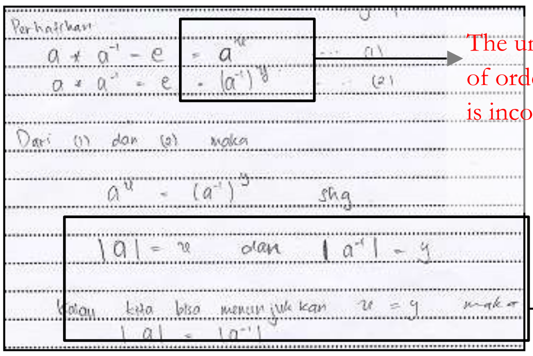
Let $|a| = m$.
It means $a^m = e$ and $\forall l < m, a^l \neq e, l \in \mathbb{N}$
We will show $|a^{-1}| = m$. ~~Since~~
~~Sin~~ It means $(a^{-1})^m = e$ and $\forall l < m, (a^{-1})^l \neq e, l \in \mathbb{N}$

Figure 10. The Improved Proof - S_3

Metacognitive Misdirection

The subject in group I, S_1 experienced a failure in metacognitive, called *misdirection* in task 2. S_1 recognized the *red flag,* no progress in proving steps about the order a and a^{-1} are equal, subsequently gave incomplete responses related to the concept of order, but still relevant. On the other hand, S_1 could recognize the *red flag* on the error detection that is done related to steps to verify the order a and a^{-1} by stating $a^x = (a^{-1})^y$ that x is the order a and y is the order a^{-1}. The subject group II, S_3 experienced a similar response in task 3 by using the concept of homomorphism that was still relevant but not appropriate in proving subgroups. Then S_4 experienced metacognitive *misdirection* on task 3 with the *red flag* error detection on the incompleteness of a subgroup owned only the closure property. This discovery is in line with metacognitive failure according to the previous research (Stillman, 2011).

Based on the information of field notes and interview session with S_1, it showed that the understanding of S_1 about the characteristics of order of an element in a group was incomplete. At first, S_1 presupposed x and y as the order a and a^{-1}, respectively. Then S_1's metacognitive activity led to an associated concept but still relevant, for example, the concept of order as a distance of a to e, such that $a^x = e$ with x the order a. Nevertheless, the second characteristic of x as the smallest positive integer such that $a^x = e$ was forgotten. Thus, it has been found that S_1 used his knowledge in the form of an incomplete corollary. Furthermore, S_1 suggested the next step prove $x = y$, but no further step towards the final result. S_1 did not respond any further to the *red flag,* and there was no more progress on his work to the right steps, in the situation metacognitive *misdirection* has raised (Stillman, 2011; Stillman, 2015). This is supported by footage statement from S_1 on the stimulated recall as follows and Figure 11.



The understanding of order's concept is incomplete

Red Flag: Finishing process is incomplete, there is no progress shown

Translation:
Note that,
$a * a^{-1} = e = a^x$ ... (1)
$a * a^{-1} = e = (a^{-1})^x$ ... (1)
From (1) and (2), then
$a^x = (a^{-1})^y$ such that
$|a| = x$ and $|a^{-1}| = y$
As if we can show $x = y$, then
$|a| = |a^{-1}|$

Figure11. Incomplete Concept - S1

S_1: Suppose we are going to prove that the order a and a^{-1} are equal –**Obj5** *(MA), so to every a^x that is the same with e then x must be equal to y where $(a^{-1})^y$ is equal to e (MA)… for this matter I thought I had to think out of abstract algebra (MA), what is order, what is distance*—**Dis**, *distance is like absolute value (MA). Ooh, I thought again maybe if it is similar in Abstract Algebra (ME), the order is the distance from a to e (MR), then related to the power of any member of a group when the value is indexed to e* –**Or** *(MR)…. But at the beginning, I had a thought that a^x is equal to $(a^{-1})^y$* –**E2** *It can be concluded that the order a is equal to the order a^{-1}* –**E3** *(MR). After thinking about it again, if it is possible (ME), evidently, I get some conditions, first is the order a is equal to x and order a^{-1} is equal to y*—**E1**, *(MA) so it cannot be done directly (MR) so I have not found the progress yet (**RED FLAG**-**Rf7**) (MA).*

Scaffolding was conducted after S_1 could complete his knowledge about the concept of order by *reviewing*, which was given by asking S_1 to pay attention to the results of his work before and looking for whether there was an error. Awakening the students' learning consciousness by recalling the previous material that has been learned (Gusmiyanti et al., 2018). Next, we continued to give scaffolding in *restructuring* by using questions to consider which concept is the most prevalent between the initial concept and the one improved concept. *Restructuring* continued by guiding on how if the complete concept of orders that have been used. All guidances are conducted following the hierarchy of scaffolding (Anghileri, 2006). S_1 could perform the appropriate and complete response after the scaffolding and increase his independence to revise his proof. The teachers express the view that clear instructions and boundaries for the students, might in fact increase their independence, since it may be easier to become independent within a more limited field (Pol et al., 2019; Zackariasson, 2019).

Scaffolding was provided in group II, especially S_3 in level *reviewing* the question of whether he realizes his failure or incomplete steps that he has done. We provided questions or commands for reflection on the proof that has been written (Basir & Wijayanti, 2020). Next, we conducted some direction on whether it is necessary to add a specific subgoal to resolve the problem with a particular concept. The guidances in exploring strategies students should do by making some subgoals to achieve the solution (Gusmiyanti et al., 2018). It provided direction in helping students to focus on achieving goals, reduce frustration, and provide clear direction on the ultimate goal of the activities undertaken (Basir & Wijayanti, 2020). Then *restructuring* was carried out through feedback questions related to the most relevant concept to the problem or maybe certain concepts that can be used, and which one is most relevant to the problem. The objective is to rebuild the initial understanding of the concept such that students can plan to solve the problem correctly (Basir & Wijayanti, 2020).

Examples of scaffolding for metacognitive *misdirection* were performed on S_4. The *reviewing* in the form of a question about the reasons of S_4 uses the concept of homomorphism and also the concept of what is not known by S_4 related to the problem (Anghileri, 2006). Asking his understanding of the problem or any element of the problem as the second *reviewing* was given. It is supported by previous research, the first stage of metacognitive scaffolding is to awaken student's awareness (Gusmiyanti et al., 2018). We continued to ask questions about the reasons of S_4 uses the concept of homomorphism. Here is the scaffolding we have conducted.

*R : When you are trying to solve this problem, what are the parts that you do not understand? **(Rev2)***

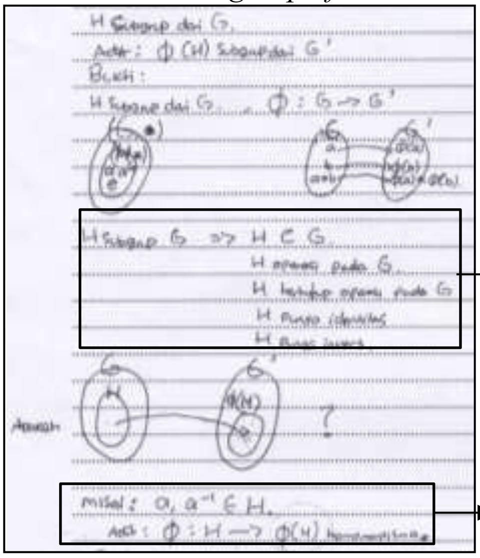
S_4: How to prove a subgroup, I forget about how to prove it, there is a theorem to prove H is the subgroup of G.

*R : What is the groundwork so that you prove it using ϕ homomorphism then $\phi(H)$ subgroup G'? Please try to explain. (**Rev1**)*

S_4: Firstly H is a subgroup G and $\phi: G \rightarrow G'$ that is a homomorphism, well, by proving like that, maybe it can prove if $\phi(H)$ is a subgroup of G'.

R : So, does it mean that you prove the subgroup by proving the homomorphism?

S_4: Yes. H is a subgroup from G, then I try if ϕ is from $H \rightarrow \phi(H)$ is a homomorphism, in fact, H is the subgroup of G.



Appropriate Concept of Subgroup

Red flag: Inappropriate Concept but relevant

Translation
H is a subgroup of G.
We will show: $\phi(H) is$ a subgroup G'.
(Diagram)
H is a subgroup of $G \Rightarrow H \subset G$
H has operation in G
H is closed to operation in G
H has an identity element
H has inverses
(Diagram)
Suppose: $a, a^{-1} \in H$
We have to show $\phi: H \rightarrow \phi(H)$ is a homomorphism.

Figure 12. One Example of Inappropriate Concept but Relevant Used by S_4

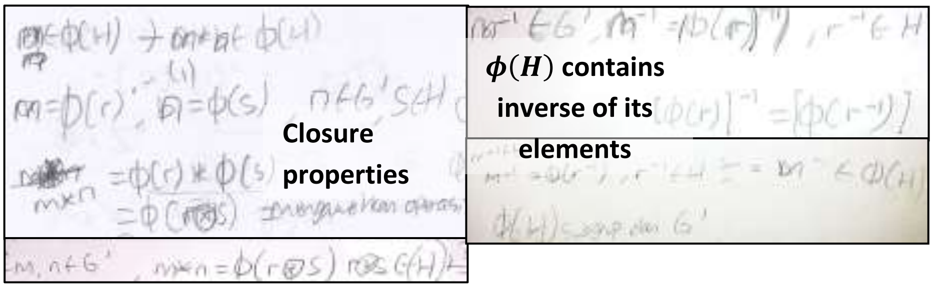


Figure 13. Improved Proof Construction Result by S_4's *Scaffolding*

Next, *restructuring* was conducted by simplifying the problem and giving questions about which concept is the most appropriate and related to the problem. We bridge to revive students' knowledge and understanding of an existing concept. Whereas building a schema means assistance in the form of a schematic/diagram that describes the problem situation, perhaps the concept of a mind map related to the problem situation (Basir & Wijayanti, 2020). Figure 12 and Figure 13 as the proof before and after scaffolding was conducted, respectively. The scaffolding process, as follows.

*R : What does the meaning of $\phi(H)$ be a subgroup of G'?(**Res**)*

S_4: It applies like this as well (pointing the meaning of H is a subgroup of G that is written on the proof construction) that $\phi(H)$ is a closed operation on G', $\phi(H)$ is associative, $\phi(H)$ has identical substance, $\phi(H)$ has an inverse. Then for this associative property, $\phi(H)$ is associative generated derived from G' that is also associative. (Writing the definition of $\phi(H)$ is the subgroup of G') there are 3 more steps.

R : If it is related to your proof, is it in line with the problem?

S_4: No, it is not, the direction of proof is also unclear.

R : Compared to the other three steps and your previous construction, which one is more suitable?

S_4: Yes, by definition is more appropriate for these three steps.

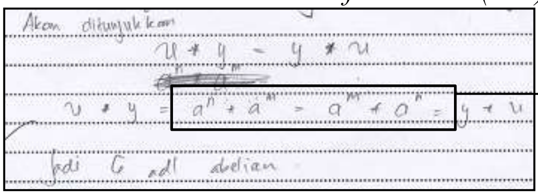
Metacognitive Vandalism

The subject in group I, S_1 experienced metacognitive *vandalism* in task 1. S_1 stuck with no progress in his solution steps, then S_1 assumed any two elements in G cyclic group, called x and y. Furthermore, by linking the cyclic group definition S_1 presupposes $x = a^n$ and $y = a^m$ with m and n are integers. S_1 briefly considered what operation was in G. After he decided to use operation $*$, S_1 could not determine the next step and encountered deadlock or impasse solution. From the *red flag* of no progress in his work, then the metacognitive activity of S_1 led him to perform the step of $a^m * a^n = a^n * a^m$ without any supported reasoning and no information on the problem. It shows the subject overcomes the deadlock then takes destructive actions by means of which students can change the problem by implementing conceptual structures improperly (Huda et al., 2018). The emergence of *vandalism* because S1 recognized the *red flags,* but the metacognitive activity directed him to use an inappropriate framework or mathematical concept (Goos, 2002; Stillman, 2011). Here are the result of the stimulated recall and Figure 14 is S_1's response with incorrect steps.

*S_1: I feel that I am in doubt in this part $a^m * a^n = a^n * a^m$(**Red flag—Rf10**) —**G3**(MA), proving $a^n * a^m$ is equal to $a^m * a^n$ (ME). If later they are equal, it means $x * y = y * x$ ... it can be concluded that G is the Abelian –**Obj7**(MA). But I had thought also that if it was a cyclic group, there must be an operation not just a regular operation like addition or multiplication (**Red flag—Rf9**) –**Gs** (MA). I thought it was an operation [in the cyclic group] but the strategy was not that easy (ME). However, I think I do it like this (MR).*

R : Later I see also before you write this down ($x = a^n, y = a^m$) you were thinking long enough, what did you think at that time?

*S_1: I was not so sure to define $x = \ a^n$ —**G2** (MA) Could I define (ME) for example I take x and y are any member of G —**G1** It was directly defined $x = a^m$ and $y = a^n$ with m and n were the member of $\mathbb{Z}$ —**G2** (MR).*



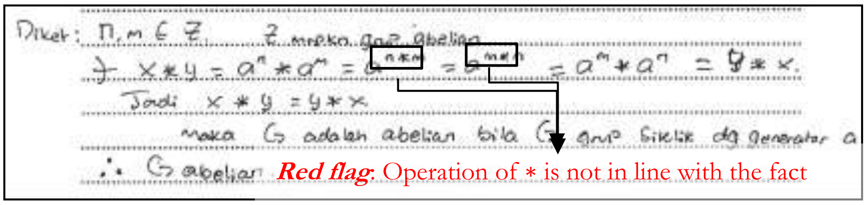
Red flag: Steps that are not in line with the problem information

Translation:
We will show
$x * y = y * x$
~~$a^n * a^m$~~
$x * y = a^n * a^m = a^m * a^n = y * x$
So, G is an abelian

Figure 14. Solution Steps are Not in Line with the Problems Condition

In subject group II, S_4 experienced a similar failure but with a different response. At the beginning of proof construction task 1, S_4 experienced an impasse solution when trying to remember the definition of a cyclic group. Then he could assume that any two elements x and y of G, appropriately, and will prove x and y satisfy the commutative properties with respect to operation in G. S_4's metacognitive activity recognized the *red flag* no progress on the solution steps, then using concepts that are inconsistent with the problem facts, e.g., "$a^m * a^n = a^{m*n}$ and $\mathbb{Z}$ is an Abelian group with * operation". He stated that $\mathbb{Z}$ commutes to operation $*$ in G. Next on task 2, S_4 could aware of the *red flag* error detection and no progress in the work steps. Even concluded without any reasoning and there was no information on the problem that supports the steps. This discovery is consistent with the emergence of *vandalism* because S_4 realized the *red flag* and still perform the work steps or inappropriate concepts (Goos, 2002; Stillman, 2011). It means the metacognitive failures marked by the noncompliance within the concept and context of the problem when responding to the *red flag* (Huda et al., 2016). On another side, we find in this research the subject does not change the condition of a problem, but they change the existing mathematical concept to be used. It differs from previous research; the students change the conditions of the problem so that the condition of the problem is under their knowledge (Goos, 2002). The following statements by S_4 related to what he was thinking when constructing proof of task 1 and Figure 15 show the answer that contains errors.

S_4: ...I thought I need to show first that x operation to y —**H1** *(MA). I tried to think like this, $x * y$ was equal to $a^n * a^m$ was equal to a^{n*m}* –**H2** *(MR). Next, $y * x$ was equal $a^m * a^n$ was equal to a^{m*n}*–**H3** *(MA), but because the advance was difficult about the cyclic group, that was the deadlock (**RED FLAG—Rf11**) (MA). That n was equal m that was the member $\mathbb{Z}$* –**Gs**, *and as I know that $\mathbb{Z}$ was an abelian group of $+$ and $\times$* –**Fc** *(MA). After I thought it again whether it guaranteed the abelian of $*$* – **Fc'** *(ME), but to achieve the goal, $x * y$ was equal to the $a^n * a^m = a^{n*m} = a^{m*n}$* –**Obj8** *(MR). Because turned to be [$\mathbb{Z}$ to $*$] abelian (**RED FLAG—Rf10**) (MA), later it can be applied of $n * m = m * n$* –**H4** *(MR).*



Translation:
Given: $n, m \in \mathbb{Z}$
$\ni x * y = a^n * a^m = a^{n*m} = a^{m*n} = a^m * a^n = y * x$
So $x * y = y * x$
Therefore G is an abelian if G is a cyclic group with a as the generator.
$\therefore G$ abelian

Figure 15. The Concept Used by S_4 is not in Line with The Mathematical Facts

On subject group I, scaffolding was conducted through *reviewing* and level 3, while subject group II was through level 2; *reviewing* and *restructuring*. *Reviewing* was performed on S_1 by asking about certain concepts or strategies that are used and the reasoning. It provides guidance for planning and guidance for monitoring (Reiser, 2004). Then we offered an explanation about cyclic group and exponent. The explanation is presented in the form of a solid statement, suitable for students understanding of what they have learned and why, when, and how to use it. (Basir & Wijayanti, 2020). Next, we continued at level 3 by discussing any alternative strategies that possibly used if the subject experienced an impasse solution. Subsequently on S_4, *reviewing* by conducting questions about what he has been known about the problem and also the reason for using inappropriate concepts or strategy before. The *restructuring* reinforced out by discussing the misconceptions that are used. These findings are in line with the hierarchy scaffolding in the previous research (Anghileri, 2006). After the scaffolding, S_1 could provide an appropriate response independently.

One example scaffolding performed on S_1, we asked a definition or theorem and the reason for its use in the taken steps. From the S_1's answers obtained that S_1 could not associate the definition with one of the theorems about the law of exponent in the group theory, that is for every a member of the group and m, n of integer then applied $a^m * a^n = a^{m+n}$. Following is the scaffolding.

*R : Try to explain what is the theorem or the definition that is related to the problem and the result that you have written? (**Rev1**)*

S$_1$: First, there are a cyclic group and generator, and also an Abelian group. If it is Abelian, the commutative properties must be applied. If I take any two members of x, y in G, then it applies that $x * y$ is equal to $y * x$. Then the generator, definition of the generator may be that a is called as the generator from that cyclic group, a is the member of G, then for any x that is member of G, there is n member of $\mathbb{Z}$ so $x = a^n$.

R : For this step (pointing $a^m * a^n = a^n * a^m$) what is the basis of your reason for doing that step?

S$_1$: Using… (thinking for a while), hang on (he cannot give the reason). It must not be like this, $*$ has not known yet what the operation is. Something missing here. When I directly conclude this, it is because I think that it is like a regular exponent. But this $*$ is different, so it cannot be done like that.

*R: Try to think again about the connection of cyclic group and exponent. (**Rev2**)*

*S$_1$: (Stops for a while and thinking) a^n is $a * a * a..* a$, it is n $-$factors. It means that the character is like a regular exponent, right. I will continue to do it for a moment. (Correcting proof construction result of $a^n * a^m = a^m * a^n$)*

Level 3 was conducted a discussion about what kind of alternative strategy could be used, and he chose the contrapositive method. Here are the excerpts of the discussion with S$_1$ as well as the result of improved response after scaffolding in Figure 16.

*R : You said that you experience an impasse in this step, have you thought about what kind of strategy that possibly used? (**L3**)*

S$_1$: Ohh, yes, maybe contrapositive.

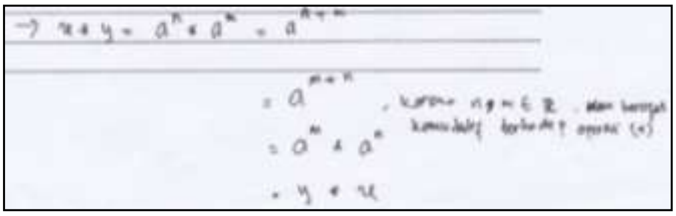
R : Can you explain what about the contrapositive generally?

S$_1$: Yes, it starts from the assumption that is not q proven that it is not p.

R : And then why do not you try to use it?

S$_1$: Before I think about how to execute it, I am focused too much on this proof.

R: For the suggestion, next time you can try every strategy that you think. Maybe if you consider it more the strategy will be easier.



Translation:
Since $n * m \in \mathbb{Z}$ and operation $+$ is commutative

Figure 16. Improved Result by S$_1$ After Scaffolding

Conclusion and Suggestion

Student's metacognitive failures in constructing mathematical proof indicate blindness, mirage, misdirection, and vandalism. Each metacognitive failure appears in different conditions. However, it is suitable for the metacognitive failure scenario with some improvisation appeared. Blindness occurs when the subject does not recognize errors detection or the ambiguity of the proof. Mirage emerges when the subject recognizes an error detection on the proper strategy or application of a theorem, then is unable to verify the truth of his work. Misdirection appears when the subject recognizes a lack of progress, then uses an incomplete-irrelevant concept. Vandalism comes out when no progress or detection of errors of the strategy then the subject performs some irrelevant steps to the issue or uses a misconception. Scaffolding level 2 and level 3 are used according to the proportion of the subject's needs until he is able to recognize red flags and successfully use the right strategy.

This research can be developed by groups of the subject or expanded to other mathematical subjects. Additionally, different questions can be used, for example, the problem can only be solved by indirect proof. It can be done in the context of metacognitive failure on various conditions, for example, associated with the difficulty of proof constructing, other problem-solving, or any characteristic of student used. Practically, the teachers can use these results for learning innovations in scaffolding-based proof courses. Metacognitive failure appears in the scenario of learning in the classroom so that further research of metacognitive failures in the social context in small groups should be expanded. It needs further investigation about characteristics from the metacognitive process or other views that occur in every metacognitive failure. To avoid metacognitive failure and especially to improve students' skills in constructing proofs, can be optimized such facilities to provide a problem of proof or validate a mathematical proof. The scaffolding might need some development and application in supporting students to overcome difficulty in proving the mathematical statement.

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